A Streamlined Nonlinear Path Following Kinematic Controller

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Outline

• Motion Control Problems of Autonomous Vehicles
• Review of Path-Following Algorithms
• A Streamlined Nonlinear Path Following Kinematic Controller
• Simulations
SISTEMA AUTONOMO PARA LA LOCALIZACION Y ACTUACION ANTE CONTAMINANTES EN EL MAR

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Motion control problems of autonomous vehicles

• Point stabilization

Design of control laws that stabilize the vehicle at a given target point with a desired orientation.
Motion control problems of autonomous vehicles

• Trajectory tracking

Design of control laws that force a vehicle to reach and follow a geometric path with an associated timing law.

Usually, tracking problems for autonomous vehicles are solved by designing control laws that make the vehicles track pre-specified feasible “state-space” trajectories, i.e., trajectories that specify the time evolution of the position, orientation, as well as the linear and angular velocities, all consistent with the vehicles’ dynamics.
Motion control problems of autonomous vehicles

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Design of control laws that force a vehicle to reach and follow a geometric path with an associated timing law.

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Motion control problems of autonomous vehicles

• Path following

Design of control laws that force a vehicle to converge to and follow a path that is specified without a temporal law.

This problem can be expressed by the following two task objectives:

• **Geometric Task**: make the position of the vehicle converge to and follow a desired geometrical path.

• **Dynamic Task**: make the vehicle satisfy a dynamic assignment along the path, e.g. the speed of the vehicle converge to and track a desired speed assignment (*maneuvering*)
The path following loop is divided in an inner control loop and an outer guidance loop.

- The inner loop controller stabilizes the vehicle dynamics.
- The outer loop controls the vehicle kinematics and computes reference commands to the inner loop controller, providing path-following capabilities.
- If there is adequate frequency separation between the guidance and control systems the combined scheme will perform as specified separately.
- This structure is the usual one when the vehicle comes equipped with an autopilot.
Path Following Algorithms

- Integrated guidance and control are designed simultaneously

A survey of control and guidance and control algorithms for vessels can be found in:

*Automática marina: una revisión desde el punto de vista del control*
J. M. de la Cruz García, J. Aranda Almansa, J.M. Girón Sierra,
Revista Iberoamericana de Automática e Informática industrial RIAI,
A waypoint path is an ordered sequence of waypoints:

\[ W = \{w_1, w_2, \ldots, w_N\}, \]

\[ w_i = (w_{n,i}, w_{e,i}, w_{d,i})^T \in R^3 \quad \text{or} \quad w_i = (w_{n,i}, w_{e,i})^T \in R^2 \]
Dubins Paths

- For a vehicle with kinematics
  \[ \dot{x} = V \cos \psi \]  \( (x, y) \) position
  \[ \dot{y} = V \sin \psi \]  \( \psi \) heading
  \[ \dot{\psi} = u, \quad u \in [-\bar{u}, \bar{u}] \]  \( V \) speed

moving at constant speed \( V \) the time-optimal path (shortest path) between two different configurations is a path formed by straight-lines and circular arc segments.

\[ R_{\text{min}} = V / \bar{u}. \]
Parametrized Path

• A parametrized path is a geometric curve $p_d(\theta)$ parametrized by a continuous path variable $\theta$.

$$ p_d(s) = [x_d(\theta), y_d(\theta), z_d(\theta)]^T \quad \text{or} \quad p_d(\theta) = [x_d(\theta), y_d(\theta)]^T $$

• Given a set of waypoints $W = \{w_1, w_2, ..., w_N\}$ a parametrized path can be generated using spline or polynomial interpolation methods.

  — Example: Cubic polynomial for $p_d(\theta) \in \mathbb{R}^2$

  $$ x_d(\theta) = a_0 + a_1 \theta + a_2 \theta^2 + a_3 \theta^3 $$
  $$ y_d(\theta) = b_0 + b_1 \theta + b_2 \theta^2 + b_3 \theta^3 $$

  Time independent path.
Parametrized Path: Reference Trajectory

- The time independent path can be transformed to a time varying trajectory by defining a speed profile along the path, $V_d(t)$,

\[
\begin{align*}
x_d'(	heta) & \triangleq \frac{dx_d(\theta)}{d\theta} \quad \rightarrow \quad \dot{x}_d(t) = x_d'(	heta) \dot{\theta}(t) \\
y_d'(	heta) & \triangleq \frac{dy_d(\theta)}{d\theta} \quad \rightarrow \quad \dot{y}_d(t) = y_d'(	heta) \dot{\theta}(t) \\
V_d(t) & = \sqrt{x_d'^2(t) + y_d'^2(t)} \quad \rightarrow \quad \dot{\theta}(t) = \frac{V_d(t)}{\sqrt{x_d'^2(\theta) + y_d'^2(\theta)}}
\end{align*}
\]

Let $s$ be the length of the path, then

\[
\begin{align*}
ds & = \sqrt{x_d'^2(\theta) + y_d'^2(\theta)} \, d\theta \\
\frac{d\theta}{ds} & = \frac{1}{\sqrt{x_d'^2(\theta) + y_d'^2(\theta)}} \\
s(t) & \triangleq \int_0^t V(\tau) \, d\tau = \int_0^t \sqrt{x_d'^2(\theta) + y_d'^2(\theta)} \, \dot{\theta}(\tau) \, d\tau
\end{align*}
\]
Simulink model for path generation

\[
\text{out} = [x_s, y_s, x_s', y_s', \psi, \kappa]
\]

\[
\psi(\theta) = a \tan \frac{y_d'(\theta)}{x_d(\theta)}
\]

\[
c(\theta) = \frac{d\psi(\theta)}{d\theta} \frac{d\theta}{ds}
\]
Outer guidance controllers for path following

- PID controllers
- Virtual Point Tracking
- Line of Sight Guidance
- A Streamlined Nonlinear Path Following Kinematic Controller
Suppose: $V$ is constant
\[ \psi_T, \dot{\psi}_T \text{ are given} \]

The control objectives are
\[ e \to 0 \]
\[ \psi_V \to \psi_T \]

Command signal:
\[ u_c = V\dot{\psi}_V \quad \Rightarrow \quad \dot{\psi}_V = u_c / V \]

PID controller
\[ u_c = V\dot{\psi}_T + K_D \dot{e} + K_P e + K_I \int_0^t e \, dt \]
if \[ \psi_V \approx \psi_T \to \dot{e} \approx V(\psi_V - \psi_T) \]

Problems
- Determine gains $K_P, K_D, K_I$ (LQR, root locus, ...) 
- Determine point $P$ (might be indeterminate)
Virtual Point Tracking: Serret-Frenet Frame

- Point $P$ defines a point on the path where a Serret-Frenet frame $\{F\}$ is defined. $\{F\}$ plays the role of a virtual point or target that should be tracked by the vehicle $Q$.

- $P$ is not the point on the path closest to $Q$ but a point that is made evolved according to a conveniently defined control law.
Virtual Point Tracking: Control Signals

**Control signals:** \( \dot{\psi}_B (V \dot{\psi}_B), \ s \)

- \( \dot{\psi}_T = \dot{s} \kappa(s) \)
- \( \psi = \psi_V - \psi_T \)
- \( \psi_V = \psi_B + \alpha \)

- \( s_1 \) along-track error, \( y_1 \) cross-track error, \( \psi \) course error
- \( s \) length that the virtual point has moved along the path
- \( \kappa(s) \) path curvature
- the path is parametrized by \( s \)
Virtual Point Tracking: Kinematic Model

Point $Q = [s_1, y_1]^T$ in $\{F\}$ evolves according to the equations

$$\dot{s}_1 = -\dot{s}(1 - y_1 \kappa(s)) + V \cos \psi$$
$$\dot{y}_1 = -\dot{s} s_1 \kappa(s) + V \sin \psi$$
$$\dot{\psi} = \dot{\psi}_V - \dot{\psi}_T = \dot{\psi}_B + \dot{\alpha} - \dot{s} \kappa(s)$$

The objective is to drive the error coordinate $(s_1, y_1, \psi)$ to zero with controls: $\dot{\psi}_B$, $\dot{s}$

Equilibrium point $(s_1, y_1, \psi) = (0, 0, 0)$

$\dot{s} = V$
Virtual Point Tracking: Kinematic Path Following Controller

\[ \dot{\psi} = \hat{\delta} - K_1 (\psi - \delta) \]  (1)

\[ \dot{s} = V \cos \psi + K_2 s_1 \]  (2)

Try to bring \( y_1 \) and \( \psi \) to 0.

(2) Try to bring \( s_1 \) to 0.

- \( \delta \) is a desired approach angle
- \( K_1, K_2 \) are design parameters
- \( \psi_B \) and \( \psi_V \) can be obtained from an IMU
Virtual Point Tracking: Approach Angle

\[ \delta \text{ can be any function of } y_1 \text{ satisfying } y_1 \delta(y_1) \leq 0 \]

\[ \delta = -\psi_\delta \tanh(2K_\delta y_1) = -\psi_d \frac{e^{2K_\delta y_1} - 1}{e^{2K_\delta y_1} + 1} \quad 0 < \psi_\delta < \pi / 2, \quad 0 < K_\delta \]

- Two design parameters: \( K_\delta \), \( \psi_\delta \)

- The equilibrium point \((s_1, y_1, \psi) = (0, 0, 0)\) is \textit{Uniformly Global Asymptotically Stable (UGAS)} and \textit{Uniformly Local Exponentially Stable (ULES)}
Line of Sight Guidance: Marine Vehicles*

- The vehicle velocity vector is directed toward a point ahead of the direct projection of the craft to the tangent, located at a distance \( \Delta > 0 \).
  “Practice of good helmsman when steering a boat”
- The approach angle is now
\[
\delta = \arctan(-y_1 / \Delta)
\]
- Since \( y_1 \delta \leq 0 \) for all \( y_1 \) the previous stability properties are kept.
- Three design parameters
\( K_1, K_2, \Delta \).
- No need to compute the curvature of the path

Line of Sight Guidance: Air Vehicles

• Originally developed for missile guidance
• Introduced by Amidi (1991) for WMR and Adopted for UAVs in Park et al. (2004, 2007)
• A reference point $P$ on the desired path at a constant distance $L_1$ is designated
• A lateral acceleration command is generated according to the direction of $P$ relative to vehicle’s velocity
Line of Sight Guidance: $L_1$ Guidance Law

- The acceleration command is equal to the centripetal acceleration required to follow a circular path that passes through the reference point and is tangent to the vehicle velocity vector.

$$L_1 = 2R \sin(\eta) \quad \rightarrow \quad a_{cmd} = \frac{V^2}{R} = \frac{2V^2}{L_1} \sin(\eta) = V \omega_v$$
Line of Sight Guidance: $L_1$ Guidance Law Properties

- The law uses instantaneous ground speed and compensates naturally for wind
- It has an element of anticipation of the desired path, enabling tight tracking of curved trajectories
- Only one parameter $L_1$ to tune.
- Lyapunov stability is proven for tracking circular paths when $L_1 < R$, and straight lines
- It approximates a PD controller when following straight-line paths
- For small perturbations when following a path, the cross track error and course error dynamics behave as a second order system

\[
\dot{y}_1 + \frac{2V}{L_1} \dot{y}_1 + \frac{2V^2}{L_1^2} y_1 = \frac{2V^2}{L_1^2} y_{\text{ref}}, \quad \psi \approx \dot{y}_1 / V
\]

- $\zeta = 0.707$
- $\omega_n = \frac{\sqrt{2}V}{L_1}$
- $\tau_L = \frac{1}{\zeta \omega_n} = \frac{L_1}{V}$

- The $L_1$ intercept can be undefined
Line of Sight Guidance: $L_1$ Guidance Law Properties

- If the control law and the natural vehicle dynamics are sufficiently faster than the guidance law, no appreciable dynamic interactions between the two schemes should be expected†.

- If this is not the case stability of the combined guidance and control law is no longer guaranteed†.

- If the dynamic of the inner control law can be characterized by a time constant $\tau_{il}$, it can be seen that the guidance system is marginally stable when $\tau_L = \tau_{il}$, so it is important to ensure $\tau_L > \tau_{il}$

  $$\tau_L > \tau_{il}$$

  A value of $\tau_L \approx 3 \tau_{il}$ or $4 \tau_{il}$ should be chosen to ensure satisfactory transient response.

- $L_1$ can be adapted to the ground speed to keep a constant $\tau_L^*$

  $$L_2 = \tau_L^* V_g$$

  with guidance law

  $$a_{cmd} = \frac{2V_g}{\tau_L^*} \sin(\eta)$$

†Papoulias, 1992, ‡Curry et al. 2013.
A Streamlined Nonlinear Guidance Law*

\[ \dot{s}_1 = -\dot{s}(1 - y_1 \kappa(s)) + V \cos \psi \]

\[ \dot{y}_1 = -\dot{s}_1 \kappa(s) + V \sin \psi \]

\[ \dot{\psi} = \dot{\psi}_V - \dot{\psi}_T = \dot{\psi}_V - \dot{s} \kappa(s) \]

Guidance Law

\[
\dot{\psi}_V = \begin{cases} 
-\frac{2V}{L} \sin(\eta), & |\eta| \leq \frac{\pi}{2} \\
-\frac{2V}{L} \text{sign}(\eta), & |\eta| > \frac{\pi}{2}
\end{cases} \quad (2a)
\]

\[ \dot{s} = V \cos \psi + K (s_1 + L) \quad (3) \]


• (2a-2b) tries to bring the cross-track error and the course error to zero

• (3), \( K > 0 \), tryes to make the vehicle follow the moving reference point with a constant along-track error \( L \).

• We do not consider a reference point on the path at a distance \( L \) from the vehicle, but a desired distance from the vehicle to the reference point on the path
Analysis of the Circular and Straight-Line Path Following

• If we consider a circular path of radius $R = \kappa(s)^{-1}$, the stationary conditions yield the relation

$$\frac{KL}{V} = \frac{1 - \cos 2\beta^*}{1 - \cos \beta^*} \quad (4)$$

$$\sin \beta^* = \frac{L}{2R}$$

• The dimensionless quantity $KL/V$ is a function of the relation $L/R$, therefore the stationary point depends only on $L$ and $R$ and not on $V$.

• (4) gives a constraint that determines $K$ adaptively as a function of the present curvature of the path, ground speed and the chosen $L$.

• If the time constant $\tau_L = L/V$ is specified then $K \in [2, 4]*1/\tau_L$. 
Straight-Line Path Following

- Stationary point
  \[ s^*_1 = -L, \quad y^*_i = 0, \quad \psi^*_i = 0, \quad \eta^*_i = 0, \quad \dot{s}^*_i = V. \]

- The equilibrium point is *Uniformly Global Asymptotically Stable* and *Uniformly Local Exponentially Stable* (Lyapunov).

- Dynamics of the along-track error
  \[ \dot{s}_1 = -K(s_1 + L). \]

- Linearizing the equations of the cross-track error and course error about de e.p.
  A second order time is obtained with
  \[ \zeta = 1/\sqrt{2}, \quad \omega_n = \frac{\sqrt{2}V}{L} \]
  and, since \( R=\infty \)
  \[ \frac{KL}{V} = 4 \quad \Rightarrow \quad K = 4 \frac{V}{L} = 2\sqrt{2} \omega_n \]
Circular Path Following: Stationary Points

\[ s_i^* = R \sin \psi^* \]
\[ y_i^* = R (1 - \cos \psi^*) \]
\[ \cos \psi^* = 1 - \frac{K}{V} (s_i^* + L) \]
\[ \sin \eta^* = -\frac{L}{2R}, \quad L < 2R \]
\[ \beta^* = \eta^* - \psi^*, \quad \eta^* = -\beta^*, \quad \psi^* = -2\beta^* \]
\[ \sin \beta^* = \sin \alpha = \frac{L}{2R}, \]
Circular Path Following: Linearized system

• Linearizing the equations of the cross-track error and course error about de e.p. a second order time is obtained with the condition that the vehicle is at a distance $L$ of the reference point.

• The linear system is exponentially stable when $0 \leq L/R \leq 1.79$

• The linear system is unstable when $1.8 \leq L/R$

• Dotted curves show the corresponding values obtained by Park et al. 2007
Circular Path Following: Domain of Attractions

**Theorem 1.** Consider the autonomous system \( \frac{dx}{dt} = f(x), \ x \in R^2 \) and let \( M \subseteq R^2 \) be a compact invariant set for the system with only one equilibrium point in its interior and no equilibrium points on the boundary. Assume that for each initial condition in \( M \) there is a unique solution, and that \( f(x) \) has continuous partial derivatives in the interior of \( M \). Let \( J \) denote the Jacobian matrix of the system. Then, if the trace of \( J \) is negative and the determinant of \( J \) is positive at the equilibrium point, the domain of attraction is either the set \( M \) or an open set \( \Omega \), whose boundary is a positively invariant periodic orbit. In the latter case, the limit set of the trajectories not in \( \Omega \) are periodic orbits.

**Corollary.** Theorem 1 tells us about the behavior when the hyperbolic equilibrium is stable. If the hyperbolic equilibrium point is unstable, then \( M \) contains at least a limit cycle.
Circular Path Following: Domain of Attractions

- The kynematic equations can be written as follows

\[
\frac{L}{V} \beta = \left( \cos \psi + \frac{KL}{V} (1 - \cos \beta) \right) \left( \frac{L}{R} - \sin \beta \right) + \sin (\psi + \beta)
\]

\[
\frac{L}{V} \psi = \begin{cases} 
-2\sin (\beta + \psi) - \frac{L}{R} \left( \cos \psi + \frac{KL}{V} (1 - \cos \beta) \right), & |\beta + \psi| \leq \frac{\pi}{2} \\
-2\text{sign} (\beta + \psi) - \frac{L}{R} \left( \cos \psi + \frac{KL}{V} (1 - \cos \beta) \right), & \frac{\pi}{2} < |\beta + \psi|
\end{cases}
\]

domain \( Q = \{ (\beta, \psi): \beta, \psi \in [-\pi, \pi] \} \)

- Three different situations are found to the equilibrium point we are analyzing
Circular Path Following: Domain of Attractions

i) \( 0 \leq L/R \leq 1.6 \) The kynematic system is UGAS and ULES

Phase portrait for \( L/R = 1, \beta^* = 30 \text{ deg}, \psi^* = -60 \text{ deg} \).

All trajectories converge to the stationary point. Blue arrows show the flow vector.
Circular Path Following: Domain of Attractions

ii) $1.6 < L/R \leq 1.79$  
The kynematic system is ULAS and ULES and the domain of attraction is a limit cycle.

Phase portrait for $L/R = 1.71$. $\beta^* = 58.76 \text{ deg}, \psi^* = -117.52 \text{ deg}$.

Some trajectories converge to the stationary point and the rest to the limit cycle.
iii) $1.7 < \frac{L}{R} < 2.0$ The equilibrium point is stable and there is a stable limit cycle

Phase portrait for $\frac{L}{R} = 1.9$. $\beta^* = 71.81$ deg, $\psi^* = -142.62$ deg.
SIMULATION: MODEL

- Kinematic model of the vehicle

\[ \dot{x} = V \cos \psi_{V_i} + w_x \]
\[ \dot{y} = V \sin \psi_{V_i} + w_y \]
\[ \ddot{\psi}_{V_i} * \tau + \dot{\psi}_{V_i} = \dot{\psi}_V \]

\( w_x, w_y \) are the components of the wind in the north and east directions, respectively. The inner loop is modeled as a first order lag with time constant \( \tau \).

In all simulations

\( V = 16 \text{ m/s} \)
\( \tau = 1 \text{ s} \)

wind with constant speed of \( 8 \text{ m/s} \)

\( L = 2 * V = 32 \text{ m} \)
\( L = 3 * V = 48 \text{ m} \)
SIMULATION: CIRCLE

Trajectory of the vehicle (green and blue) and the reference point (red)

Control signals: \( \psi_V \) (deg/s) in blue, and \( ds/dt \) (m/s) in red
When $L = 3V$ the mean following error when the circle has been reached is $1.0\ m$ with standard deviation $1.2\ m$. 
SIMULATION: Parameterized Curve

Trajectory of the vehicle (black) and the reference point (red)

Control signals: $\psi_v \ (\text{deg/s})$ in blue, and $ds/dt \ (\text{m/s})$ in red

Maximum separation error at the curves: $0.5 \ m, 1.5 \ m, 4 \ m$ for $L = 2V, 4V, 6V$. 
Applications
A Streamlined Nonlinear Path Following Kinematic Controller

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